12 Math Rules That EXPIRE in the MIDDLE GRADES

Turn away from overgeneralizations and consider alternative terminology and notation to support student understanding.

Karen S. Karp, Sarah B. Bush, and Barbara J. Dougherty

Many rules taught in mathematics classrooms "expire" when students develop knowledge that is more sophisticated, such as using new number systems. For example, in elementary grades, students are sometimes taught that "addition makes bigger" or "subtraction makes smaller" when learning to compute with whole numbers, only to find that these rules expire when they begin computing with integers (Karp, Bush, and Dougherty 2014). However, middle-grades students, especially those who are struggling, often try to force-fit the rules that they remember from the elementary grades to new concepts or skills.

In this article, we present 12 persistent rules that expire. These are "rules"

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Table 1 Students' analyses of reasoning provide illuminating information.		
Equation Chosen as Easy	**Student Reasoning**	**Percentage of Students Choosing the Equation (n = 50)**
$-x + (5) = 8$	• "I chose this because I can rewrite the equation as $5 - x = 8$. That's easier to solve." • "It looks normal like stuff I did in first grade."	36
$-6 = 3(x + 1)$	• "I don't like this one 'cause you have to do parentheses." • "My teacher last year said that you have to flip this one before you can do anything because the letter has to be on the left. I don't like doing that."	18
$3x + 8 = -10$	• "I picked this one because the letter is on the left and it's supposed to be."	44
$-8 = 2x + 20$	• "You just have to turn this one around and then it's easy. You gotta make sure the letter is on the left. I don't know why math teachers put letters on the right."	2

that we have found prevalent in our many years of working with students, from mathematics education literature, or in some cases, rules that we ourselves have taught and later regretted. In each case, we offer mathematically correct and more helpful alternatives.

The Common Core's Standards for Mathematical Practice (SMP) encourage precision, including the appropriate use of mathematics vocabulary and notation and the reasoned application of "rules" (CCSSI 2010). This leads to instruction focused on sense making and reasoning—the very experience described in NCTM's *Principles to Actions: Ensuring Mathematical Success for All* (2014).

GETTING STARTED

Imagine this scenario in your mathematics classroom, in which you present the following set of one-variable equations:

1. $-x + (5) = 8$
2. $-6 = 3(x + 1)$
3. $3x + 8 = -10$
4. $-8 = 2x + 20$

Which of these equations would your students choose to solve first or find the easiest? In our work, we found that students were comfortable solving equations 1 and 3 because they "looked normal" with the "operation" first (on the left side), followed by the "answer" (on the right side). Students often hesitated at equations that were similar to 2 and 4 because the perceived "operation" and "answer" were arranged in a seemingly reversed order (see **table 1**). This inflexibility can be linked to the idea that middle school students often have yet to interpret and understand the equal sign as a symbol indicating a relationship between two quantities (Mann 2004). Additionally, students may think that the solution to an equation always goes on the right side of the equal sign. These overgeneralizations are not helpful and can have a negative impact on students' conceptual understanding. We suggest that these students are experiencing rules that expire (Karp, Bush, and Dougherty 2014).

We highlight rules sometimes used with middle school students that seem to hold true at the moment, given the content the student is learning at that time. However, students will later find that these rules expire. Sometimes taught as shortcuts with content that students learned in the previous grades, these rules expire when students use them inappropriately with more advanced problems and find that they are incorrect. Such experiences can be frustrating and can promote the belief that mathematics is a mysterious set of tricks and tips to memorize rather than concepts that relate to one another. For each rule that expires, we do the following (similar to Karp, Bush, and Dougherty 2014):

1. State the rule.
2. Discuss how students overgeneralize it.
3. Provide counterexamples.
4. State the expiration date or the point when the rule begins to fall apart.

12 RULES THAT EXPIRE

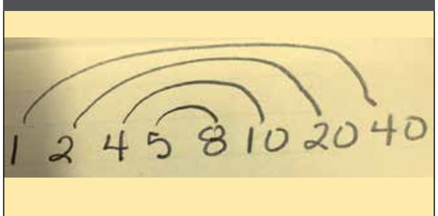
1. *KFC: Keep-Flip-Change.* When learning to divide fractions, students are sometimes taught to KFC (Keep-Flip-Change) or told "Yours is not to reason why, just invert and multiply." Although both versions align with the standard algorithm, students might overgeneralize

“My teacher last year said that you have to flip this one before you can do anything because the letter has to be on the left. I don’t like doing that.”

this rule to other operations with fractions. Additionally, these mnemonics and sayings do not promote conceptual understanding, making it challenging for students to apply them in a problem-solving context. Instead, division of fractions can be linked to whole-number division by asking how many groups of the divisor make up the dividend. Although students will eventually use the algorithm, they should gain a conceptual understanding of dividing fractions through the use of physical models (Cramer et al. 2010) or other methods, such as the common denominator strategy.
Expiration date: Grade 6 (6.NS.1)

2. *Use the factor rainbow to factor.*
Sometimes students are taught to create a “factor rainbow” to list all factors of a number. For example, if finding the factors of 20, students would write 1 and 20, then 2 and 10, and 4 and 5 (see **fig. 1**). The rule is taught so that once you identify factors that are consecutive numbers (e.g., 4 and 5), you have identified all factors. However, factors can be missed with this approach. Some numbers do not have consecutive factor pairs, as in when finding the factors of 40: 5 and 8 are the closest factors. Moreover, this rule expires when factoring a square number (its factors are not all paired) or when working with non-whole-number factors, such as in algebra when students factor out fractional parts as well as variables.
Expiration date: Grade 6 (6.NS.4)

Fig. 1 Drawing a factor rainbow for 40 is one way to visualize the process.



3. *The absolute value is just the number.*
Students are sometimes told that the absolute value of a number is that number, with a positive sign. For example, $|-4| = 4$ because you drop the negative sign. Confusion sets in when students are presented with $-|-4|$ because they are unsure what this represents. How can absolute value be negative? Without making sense of the meaning of absolute value (that is, its distance from zero on a number line), students may not interpret it correctly within particular contexts.
Expiration date: Grade 6 (6.NS.7)

4. *Multiplication is repeated addition.*
Considering multiplication as only repeated addition can result in students thinking that the expression 3^3 is equivalent to $3 + 3 + 3$. This thinking leads to overgeneralizations because students come to believe that 3 raised to the third power means that 3 is used as an addend 3 times. Writing such expressions in correct expanded form can help with this misunderstanding.
Expiration date: Grade 6 (6.EE.1)

5. *PEMDAS: Please Excuse My Dear Aunt Sally.*
This mnemonic phrase is sometimes taught when students solve numerical expressions involving multiple operations. At least three overgeneralizations commonly occur with this rule:

- Students incorrectly believe that they should always do multiplication before division, and addition before subtraction, because of their order in the mnemonic (Linchevski and Livneh 1999), instead of performing them in the order in which they appear in the expression.
- Students perceive that the order of PEMDAS is rigid. For example, in the expression

  $$30 - 4(3 + 8) + 9 \div 3,$$

  there are options as to where to begin. Students actually have a choice and may first simplify the $3 + 8$ in the parentheses, distribute the 4 to the 3 and to the 8, or perform $9 \div 3$ before doing any other computation—all without affecting an accurate outcome.
- The P in PEMDAS suggests that parentheses are first, but this should also represent other grouping symbols, including brackets, braces, square root symbols, and the horizontal fraction bar. We suggest making sense of a problem. However, if using a hierarchical model, consider this order: (a) Grouping symbols or exponents; (b) multiplication or division; and (c) addition or subtraction.
  Expiration date: Grade 6 (6.EE.2)

6. *A solution to an equation must be in the form $x = \square$.*
Students are often taught that the variable and/or operation comes first, followed by the answer (e.g., the constant) in an algebraic equation

(Dougherty and Foegen 2011). However, this rule has no mathematical necessity because the equal sign indicates that two quantities are equivalent. Therefore, variables, operations, and constants can be located on either or both sides of the equal sign. Instead of overgeneralizing that an equation should "look" a certain way, we as teachers should promote flexibility in students' thinking. When the teacher uses a specified set of steps and the placement of the solution in that format, students lose sight of the conceptual aspects of equations and instead focus on implementing algorithmic steps.
Expiration date: Grade 6 (6.EE.4)

7. *The "Butterfly Method" for comparing fractions.*
Students are frequently taught the "Butterfly Method," which refers to cross multiplying two fractions to determine which fraction is greater. For example, in

$$\text{Which is greater, } \frac{3}{4} \text{ or } \frac{7}{8}?$$

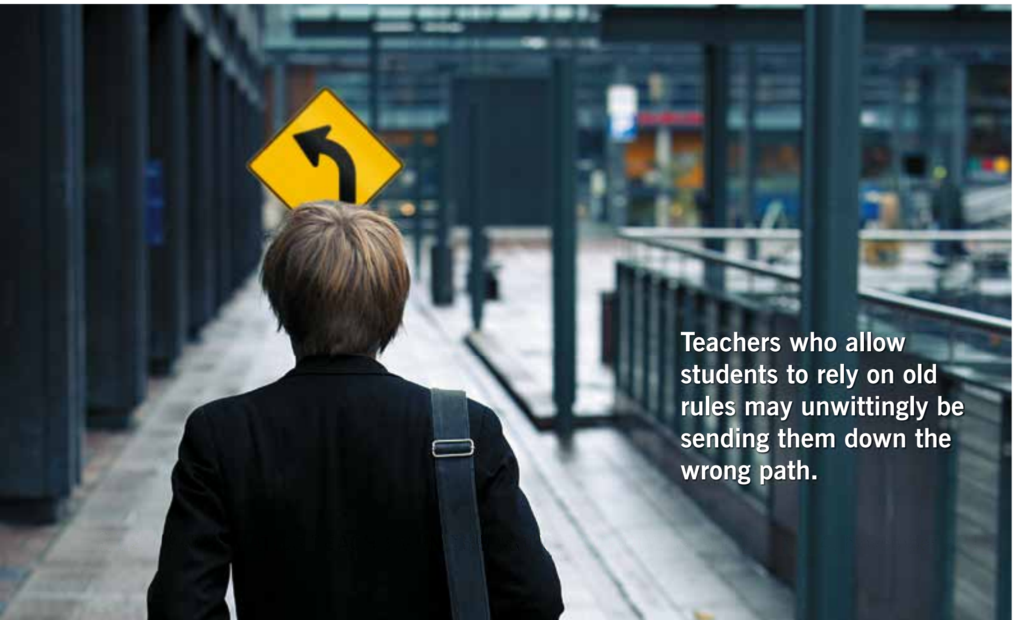
students may draw a loop around the 3 and 8 and around the 4 and 7 (which looks like a butterfly) and multiply $3 \times 8 = 24$ and $4 \times 7 = 28$ to determine that 24 is less than 28, which tells them that

$$\frac{3}{4} \text{ is less than } \frac{7}{8}.$$

However, this rule is problematic for several reasons. First, it does not foster conceptual understanding of the numerical value of fractions because it removes the need to understand the relationship between the two fractions or consider the quantities they represent. Second, students begin to overgeneralize and incorrectly apply this rule to other situations whenever they see two fractions, such as when they add, subtract, multiply, or divide fractions.
Expiration date: Grade 7 (7.RP.2)

8. *The most you can have is 100 percent of something.*
Students are sometimes taught that because 100 percent is equivalent to 1 whole, that is the most they can have. However, increases and decreases can be of any size, including more than 100 percent. This rule expires as students work with ratios and proportional relationships involving markups, discounts, commissions, and so on. *Expiration date:* Grade 7 (7.RP.3)

9. *Two negatives make a positive.*
This rule may be taught when students learn about multiplication and division of integers and is used to help students quickly determine the sign of the product or quotient. However, this rule does not always hold true for addition and subtraction of integers, such as in $-5 + (-3) = -8$. Additionally, this rule does not foster the understanding of why the product or quotient of two or more integers is negative or positive. Instead of focusing on the rule, consider using patterns of products to develop generalizations about the relationship between factors and products.
Expiration date: Grade 7 (7.NS.2)



Teachers who allow students to rely on old rules may unwittingly be sending them down the wrong path.

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10. *Use keywords to solve word problems.*
A keyword approach is frequently introduced in the elementary grades and extends throughout a student's school career as a way to simplify the process of solving word problems. However, using keywords encourages students to overgeneralize by stripping numbers from the problem and using them to perform a computation outside the problem context (Clement and Bernhard 2005). This removes the act of making sense of the actual problem from the process of solving word problems. Many keywords are common English words that can be used in many different ways. Often a list of words and corresponding operations are given so that word problems can be translated into a symbolic, computational form. For example, students are told that if they see the word *of* in a problem, they should multiply all the numbers given in the problem. Likewise, although the keyword *quantity* sometimes signifies the need for the distributive property, at other times it does not. Keywords are especially troublesome in the middle grades as students explore multistep word problems and must decide which keywords work with which component of the problem. Although keywords can be informative, using them in conjunction with all other words in the problem is critical to grasping the full meaning.
Expiration date: Grade 7 (7.NS.3)

Another student analyzed –8 = 2x + 20: "You just have to turn this one around and then it's easy. You gotta make sure the letter is on the left. I don't know why math teachers put letters on the right."

11. ***A variable represents a specific unknown.***
When students work with one-variable equations, the solutions to the equations are almost always one specific value (e.g., $x = 5$ or $x = -3$). However, students overgeneralize this as being true for all situations involving variables, yet this rule quickly expires as variables take on other meanings, such as varying quantities or parameters (e.g., $y = mx + b$), labels ($A = bh$), or generalized unknowns. Additionally, students may not accept equations that represent identities (such that the variable can take on *any* value) or equations that have no solution (such as $3x + 4 = 3x - 4$). This rule expires when students begin to work with linear functions.
Expiration date: Grade 8 (8.EE.7)

12. ***FOIL: First, Outer, Inner, Last.***
When learning to multiply two binomial expressions, students might be taught to FOIL, that is, to multiply the first term in the first binomial by the first term in the second, then multiply the outer terms of each binomial, then the inner terms of each binomial, and then the second (last) terms of each binomial. Although this rule works for binomials, it soon expires as students begin multiplying other polynomials, such as a binomial and a trinomial, or two trinomials. Instead, have students explore how they are really using the distributive property multiple times, to multiply each term in one polynomial by each term in the other polynomial.
Expiration date: High school (A.APR.1)

EXPIRED LANGUAGE AND NOTATION

We must also consider the mathematical language and notation that we use and that we allow our students to use. The ways in which we communicate about mathematics may bring with them connotations that result in students' misconceptions or misuses, many of which relate to the previously discussed Rules That Expire. Using terminology and notation that are accurate and precise (SMP 6) develops student understanding that withstands the growing complexity of the secondary grades. **Table 2** includes commonly used expired language and notation, gathered from our years of experience in the classroom, paired with alternatives that are more appropriate.

"BUT MS. JONES SAID SO"

Coherence is one of the major emphases in CCSSM. By having a series of rigorous standards at each grade, with less overlap and structured alignment, students can progress more purposefully through the content. By building a schoolwide plan for the consistent and precise presentation of rules, terminology, and notation used by all teachers, students will never get caught in the "But Ms. Jones said so" mode of finding something in their past instruction that is no longer accurate. Through such intentional consistency, students are able to focus on the new ideas presented as the language and tenets continue to be the foundation for lessons. Because the middle-grades years are pivotal in cementing the ideas from elementary school and building the concepts needed for high school, this explicit, systemic consistency is critical. As we avoid these 12 Rules That Expire, we instead find ways to present a seamless and logical world of mathematical ideas.

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Are There Other Rules That Expire?

We invite *MTMS* readers to submit additional instances of "rules that expire" or "expired language" that this article does not address. Join us as we continue this conversation on *MTMS*'s blog at **www.nctm.org/12rules**, or send your suggestions and thoughts to mtms@nctm.org. We look forward to your input.

Table 2 These alternatives can be used in place of expired language and/or notation.

What Is Stated and/or Notated	Alternative Appropriate Statements or Notations
Using the notation $8 + 4 = 12 + 5 = 17 + 3 = 20$ to symbolize a series of addition problems	Stringing together a series of additions (or other computations) cannot be connected with equal signs, as the components are unequal. Instead, use individual equations, each using the answer of the previous problem as the starting addend. Equal signs must connect equal quantities.
Using a diagonal bar in fraction notation	This notation becomes problematic with polynomials and for learners who often read the handwritten diagonal as a 1 (e.g., 3/4 is read as 314). Use a horizontal bar instead. For 1/2, write $\frac{1}{2}$.
Getting rid of the fraction or decimal	Students create an equivalent equation by multiplying or dividing and are not doing away with the fraction or decimal point at all. For example, $\frac{1}{2}x + 4 = \frac{1}{4}$ becomes $2x + 16 = 1$ by multiplying each term by 4.
Using *rounding* to mean the same as *estimating* Using *guess* to mean the same as *estimate*	An *estimate* is an educated approximation of a calculation of an amount of a given quantity. It is not a random guess. Rounding is one strategy to produce a computational estimate, but it is not synonymous with an estimate.
Using *point* to read a decimal, such as “three point four” for 3.4	Instead, read a decimal as a fraction: 3.4 is “three and four-tenths.” This will make converting decimals into fractions an easier task. Use the word *point* only when describing how a decimal is written or in a geometric context.
Reducing fractions	Using the term *reducing* may cause students to think the fraction value is getting smaller. Instead, use the term *simplifying fractions*, or instruct students to *write the fraction in simplest form or lowest term*s.
Plugging in a value for a variable	*Plugging in* is not a mathematical term. Instead, students should *substitute* a value.
Saying that fractions have a *top and bottom number*	A fraction is one number, one value. The *numerator* and *denominator* should be used to describe where different digits of a fraction are located. The words *top* and *bottom* have no mathematical meaning and may incorrectly imply that a fraction consists of more than one number.
Using the first letter of the word to describe the variable	For example, if you use the variable c to represent the number of cars in a problem, when students see $4c$ in the equation, they think it means 4 cars (using c as a label) rather than 4 times the number of cars. When you select a variable, avoid the first letter of the word and use instead an arbitrary letter to represent the *number* of cars.
Moving the decimal point when dividing decimals	The decimal point does not actually move. Rather, the digits are shifted when an alternative equation is made by changing the divisor and the dividend by multiplying (or dividing) both by a power of 10.

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Any thoughts on this article? Send an email to **mtms@nctm.org**.—*Ed.*

Karen S. Karp, kkarp1@jhu.edu, is a visiting professor at Johns Hopkins University in Baltimore, Maryland. She is professor emeritus at the University of Louisville in Kentucky, a past member of the NCTM Board of Directors, and a former president of the Association of Mathematics Teacher Educators. Her current scholarship focuses on teaching interventions for students in the elementary and middle grades who are struggling to learn mathematics. **Sarah B. Bush**, sbush@bellarmine.edu, an associate professor of mathematics education at Bellarmine University in Louisville, Kentucky, is a former middle-grades math teacher who is interested in relevant and engaging middle-grades math activities. **Barbara J. Dougherty**, barbdougherty32@icloud.com, a research professor for mathematics education at the University of Missouri–Columbia, is a past member of the NCTM Board of Directors and the editor for the Putting Essential Understandings into Practice series. She is a co-author of conceptual assessments for progress monitoring in algebra and curriculum modules for middle school interventions for students who struggle.